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ELECTROELASTIC EQUILIBRIUM OF A THIN ANISOTROPIC LAYER WITH PIEZOELECTRIC EFFECTS TAKEN INTO ACCOUNT

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The method of asymptotic integration [1] of three-dimensional equations of electroelasticity is used to construct a solution for the general cale of anisotropy of the problem of equilibrium of a thin piezoelectric layer the plane edges of which are free of electrode coating and are acted upon by specified, external mechanical forces. A fundamental iterative process is constructed, which makes possible the estimation of the validity of various simplifying assumptions made in the course of constructing the practical theories [2-4]. It is established that the absence of a plane of material symmetry within the layer parallel to its middle surface does not affect the construction of the first iterative step, but affects the derivation of the relations in the subsequent approximations.

1. Let us consider a thin piezoelectric layer of constant thickness $2h$. We assume that the mass forces and electrical volume charges are absent from the layer and the external physical forces are specified on the plane edges of the layer. The magnetizability of the layer is neglected.

We adopt the undeformed middle plane of the layer as the x_1x_2 coordinate plane, $-h \leqslant x_3 \leqslant h$, and assume that the field of deformations appearing in the layer is characterized by the vector $\mathbf{R}$ $(r_{11}, r_{22}, r_{33}, r_{23}, r_{13}, r_{12})$, the stress field by the vector $\mathbf{T}$ $(t_{11}, t_{22}, t_{33}, t_{23}, t_{13}, t_{12})$, and the displacement field by the vector $\mathbf{u}$ (u_1, u_2, u_3). The electrostatic field appearing in the layer is characterized by the potential $\mathbf{E}$ (E_1, E_2, E_3) and electrical induction $\mathbf{D}$ (D_1, D_2, D_3) vectors. Let us denote by v the electric field potential, $E_m = \partial v / \partial x_m$, and introduce the dimensionless quantities

$$\xi_1 = \frac{x_1}{a}, \quad \xi_2 = \frac{x_2}{a}, \quad \xi_3 = \frac{x_3}{h}, \quad \lambda = \frac{h}{a} \tag{1.1}$$

where a is a linear parameter. We shall call the layer thin, if its dimensionless half-thickness λ is less than one. The equations of electrostatic equilibrium assume the form [5, 6]

$$\partial_1 t_{1i} + \partial_2 t_{i2} + \frac{1}{\lambda}\partial_3 t_{i3} = 0, \quad \partial_1 D_1 + \partial_2 D_2 + \frac{1}{\lambda}\partial_3 D_3 = 0 \tag{1.2}$$

$$(\partial_i = \partial / \partial \xi_i, \ i = 1, 2, 3)$$

The thermodynamic relations connecting the mechanical and electrical characteristics of the layer in a linear manner can be written in the form [1]

$$\mathbf{R} = S\mathbf{T} + \frac{1}{4\pi} G\mathbf{D}, \quad \mathbf{E} = -G'\mathbf{T} + B\mathbf{D} \tag{1.3}$$

Here S denotes the symmetric matrix of the flexibility moduli $s_{ij}{}^D$ measured at a constant value of electric induction, $\mathbf{B}$ is a symmetric matrix of the coefficients of dielectric susceptibility $\beta_{ij}{}^t$ measured under constant mechanical stresses, G is, in general, a nonsymmetric matrix of piezoelectric moduli g_{ij} and a prime denotes transposition .

We shall assume that at least one of the electromechanical constants $s_{i4}{}^D$, $s_{i5}{}^D$, g_{3i} $(i = 1, 2, 3, 6)$, g_{j4} g_{j5}, $\beta_{j3}{}^t$ $(j = 1,2)$ is different from zero, i.e. we shall consider a general case of anisotropy of the layer material.

Assume that the plane edges of the layer are not covered by the electrodes and, that external mechanical forces are specified at these edges, i.e.

$$t_{i3} = q_{\pm i}, \quad D_3 = 0, \quad \xi_3 = \pm 1 \tag{1.4}$$

where $q_{\pm i}$ (ξ_1, ξ_2) are known functions. When the plane edges of the layer are covered by the electrodes, the relations (1.3) must be written in a different form.

2. Let us now construct the basic iterative process [1] to which the solution of the problem (1.2)—(1.4) can be reduced. Without affecting the generality of the investigation, we can assume that

$$q_{\pm i} = \sum_{n=0}^{\infty} \lambda^n q_{\pm i}^{(n)} \tag{2.1}$$

Let us denote by P any characteristics of the electroelastic state of the layer. As in [1], we define it in the form of a series

$$P = \lambda^m \sum_{n=0}^{\infty} \lambda^n P^{(n)} \tag{2.2}$$

The index m assumes integral values, characteristic for each quantity. The values must be chosen in such a manner, that substitution of the expressions of the form (2.2) into the relations (1.2) —(1.4) and equating the expressions accompanying like powers of λ yields a noncontradictory sequence of sets of equations for determining the coefficients of (2.2) with nontrivial solutions. We shall call such values of m noncontradictory. Taking into account the assumptions (2.1), we find that

$m = -3$ for $a^{-1}u_3$;

$m = -2$ for $a^{-1}u_1$, $a^{-1}u_2$, $a^{-1}v$, t_{11}, t_{22}, t_{12}, D_1, D_2;

$m = -1$ for t_{13}, t_{23}, D_3;

$m = 0$ for t_{33}.

The recurrent system of differential equations and the corresponding boundary conditions assume the form

$$\partial_3 u_3^{(n)} = \mathbf{A}_{11}\mathbf{X}_2^{(n-2)} + \mathbf{A}_{12}\mathbf{X}_3^{(n-3)} + s_{33}^D t_{33}^{(n-4)} \tag{2.3}$$

$$\partial_3 \mathbf{X}_1^{(n)} = -\mathbf{M}_1 u_3^{(n)} + B_{11}\mathbf{X}_2^{(n-1)} + B_{12}\mathbf{X}_3^{(n-2)} + \mathbf{A}_{21} t_{33}^{(n-3)}$$

$$B_{21}\mathbf{X}_2^{(n)} = M_2 \mathbf{X}_1^{(n)} - B_{22}\mathbf{X}_3^{(n-1)} - \mathbf{A}_{31} t_{33}^{(n-2)}$$

$$\partial_3 \mathbf{X}_3^{(n)} = -M_2' \mathbf{X}_2^{(n)}, \quad \partial_3 t_{33}^{(n)} = -\mathbf{M}_1 \mathbf{X}_3^{(n)}$$

$$\mathbf{X}_3^{(n)} = \mathbf{Q}_{\pm}^{(n-1)}, \quad t_{33}^{(n)} = q_{\pm 3}^{(n)}, \quad \xi_3 = \pm 1$$

where

$$\mathbf{X}_1^{(n)} = (u_1^{(n)}, u_2^{(n)}, v^{(n)}), \quad \mathbf{X}_2^{(n)} = (t_{11}^{(n)}, t_{22}^{(n)}, t_{12}^{(n)}, D_1^{(n)}, D_2^{(n)})$$

$$\mathbf{X}_3^{(n)} = (t_{13}^{(n)}, t_{23}^{(n)}, D_3^{(n)}), \quad \mathbf{Q}_{\pm}^{(n)} = (q_{\pm 1}^{(n)}, q_{\pm 2}^{(n)}, 0)$$

$$\mathbf{A}_{11} = \left(s_{13}^D, s_{23}^D, s_{36}^D, \frac{1}{4\pi} g_{13}, \frac{1}{4\pi} g_{23}\right)$$

$$\mathbf{A}_{12} = \left(s_{35}^D, s_{31}^D, \frac{1}{4\pi} g_{33}\right), \quad \mathbf{A}_{21} = (s_{35}^D, s_{34}^D, -g_{33})$$

$$\mathbf{A}_{31} = (s_{13}^D, s_{23}^D, s_{36}^D, -g_{13}, -g_{23}), \quad \mathbf{M}_1 = (\partial_1, \partial_2, 0)$$

$$B_{11} = \begin{Vmatrix} s_{15}^D & s_{25}^D & s_{56}^D & \frac{1}{4\pi} g_{15} & \frac{1}{4\pi} g_{25} \\ s_{14}^D & s_{24}^D & s_{46}^D & \frac{1}{4\pi} g_{14} & \frac{1}{4\pi} g_{24} \\ -g_{31} & -g_{32} & -g_{36} & \beta_{13}^t & \beta_{23}^t \end{Vmatrix}$$

$$B_{12} = \begin{Vmatrix} s_{45}^D & s_{44}^D & \frac{1}{4\pi} g_{34} \\ s_{55}^D & s_{45}^D & \frac{1}{4\pi} g_{35} \\ -g_{35} & -g_{34} & \beta_{33}^t \end{Vmatrix}$$

$$B_{21} = \begin{Vmatrix} s_{11}^D & s_{12}^D & s_{16}^D & \frac{1}{4\pi} g_{11} & \frac{1}{4\pi} g_{21} \\ s_{12}^D & s_{22}^D & s_{26}^D & \frac{1}{4\pi} g_{12} & \frac{1}{4\pi} g_{22} \\ s_{16}^D & s_{26}^D & s_{66}^D & \frac{1}{4\pi} g_{16} & \frac{1}{4\pi} g_{26} \\ -g_{11} & -g_{12} & -g_{16} & \beta_{11}^t & \beta_{12}^t \\ -g_{21} & -g_{22} & -g_{26} & \beta_{12}^t & \beta_{22}^t \end{Vmatrix}$$

$$B_{22}' = \begin{Vmatrix} s_{15}^D & s_{25}^D & s_{56}^D & -g_{15} & -g_{25} \\ s_{14}^D & s_{24}^D & s_{46}^D & -g_{14} & -g_{24} \\ \frac{1}{4\pi} g_{31} & \frac{1}{4\pi} g_{32} & \frac{1}{4\pi} g_{36} & \beta_{13}^t & \beta_{23}^t \end{Vmatrix}$$

$$M_2' = \begin{Vmatrix} \partial_1 & 0 & \partial_2 & 0 & 0 \\ 0 & \partial_2 & \partial_1 & 0 & 0 \\ 0 & 0 & 0 & \partial_1 & \partial_2 \end{Vmatrix}$$

For practical application of the iterative process (2.3) we use the method given in [1]. For the first two approximations ($n = 0,1$) we have

$$u_3^{(n)} = u_3^{(n,0)}(\xi_1, \xi_2) \tag{2.4}$$
$$\mathbf{X}_1^{(n)} = \mathbf{X}_1^{(n,0)} - \xi_3 \mathbf{M}_1 u_3^{(n,0)} + {}^1/_2 (1 - \xi_3^2) B_{11} B_{21}^{-1} M_2 \mathbf{M}_1 u_3^{(n-1,0)}$$
$$\mathbf{X}_2^{(n)} = B_{21}^{-1} M_2 \mathbf{X}_1^{(n,0)} - \xi_3 B_{21}^{-1} M_2 \mathbf{M}_1 u_3^{(n,0)} + {}^1/_2 (1 - \xi_3^2)\, \mathbf{X}_2^{(n,1)}$$
$$\mathbf{X}_2^{(n,1)} = B_{21}^{-1} (M_2 B_{11} + B_{12} M_2') B_{21}^{-1} M_2 \mathbf{M}_1 u_3^{(n-1,0)}$$
$$\mathbf{X}_3^{(n)} = {}^1/_2 (1 + \xi_3)\, \mathbf{Q}_+^{(n-1)} + {}^1/_2 (1 - \xi_3)\, \mathbf{Q}_-^{(n-1)} -$$
$${}^1/_2 (1 - \xi_3^2) M_2' B_{21}^{-1} M_2 \mathbf{M}_1 u_3^{(n,0)} - {}^1/_6 (\xi_3 - \xi_3^3) M_2' \mathbf{X}_2^{(n,1)}$$
$$t_{33}^{(n)} = {}^1/_2 (q_3^{(n)} + q_{-3}^{(n)}) + {}^3/_4 (\xi_3 - {}^1/_3 \xi_3^3)(q_3^{(n)} - q_{-3}^{(n)}) +$$
$${}^1/_4 (1 - \xi_3^2)\, \mathbf{M}_1 (\mathbf{Q}_+^{(n-1)} - \mathbf{Q}_-^{(n-1)}) + {}^1/_4 (\xi_3 - \xi_3^3)\, \mathbf{M}_1 (\mathbf{Q}_+^{(n-1)} +$$
$$\mathbf{Q}_-^{(n-1)}) - {}^1/_{24} (1 - \xi_3^2)^2 \mathbf{M}_1 M_2' \mathbf{X}_2^{(n,1)}$$

The system of differential equations for the unknown $\mathbf{X}_1^{(n,0)}\ (u_1^{(n,0)},\ u_2^{(n,0)},\ v^{(n,0)})$ and $u_3^{(n,0)}$ has the form

$$M_2' B_{21}^{-1} M_2 \mathbf{X}_1^{(n,0)} = -{}^1/_2 (\mathbf{Q}_+^{(n-1)} - \mathbf{Q}_-^{(n-1)}) + {}^1/_3 M_2' \mathbf{X}_2^{(n,1)} \tag{2.5}$$
$$\mathbf{M}_1 M_2' B_{21}^{-1} M_2 \mathbf{M}_1 u_3^{(n,0)} = {}^3/_2 [q_3^{(n)} - q_{-3}^{(n)} + \mathbf{M}_1 (\mathbf{Q}_+^{(n-1)} + \mathbf{Q}_-^{(n-1)})]$$

When $n < 0$, we have (2.3) –(2.5) $P^{(n)} \equiv 0$

3. Let us now consider a thin piezoelectric plate the plane edges of which are free of physical forces, with the factors deforming the plate specified on its side surface. In this case we have

$$\mathbf{X}_1^{(n)} = \mathbf{X}_1^{(n,0)} - \xi_3 (\mathbf{M}_1 u_3^{(n,0)} - B_{11} M_3 \mathbf{\Phi}^{(n-1)}) + \tag{3.1}$$
$${}^1/_2 (1 - \xi_3^2) B_{11} B_{21}^{-1} M_2 \mathbf{M}_1 u_3^{(n-1,0)}$$
$$\mathbf{X}_2^{(n)} = M_3 \mathbf{\Phi}^{(a)} + B_{21}^{-1} M_2 \mathbf{X}_{1,0}^{(n,0)} - \xi_3 B_{21}^{-1} M_2 (\mathbf{M}_1 u_3^{(n,0)} -$$
$$B_{11} M_3 \mathbf{\Phi}^{(n-1)}) + {}^1/_2 (1 - \xi_3^2)\, \mathbf{X}_2^{(n,1)}$$
$$\mathbf{X}_3^{(n)} = -{}^1/_2 (1 - \xi_3^2) M_2' B_{21}^{-1} M_2 (\mathbf{M}_1 u_3^{(n,0)} - B_{11} M_3 \mathbf{\Phi}^{(n-1)}) -$$
$${}^1/_6 (\xi_3 - \xi_3^3) M_2' \mathbf{X}_2^{(n,1)}$$
$$t_{33}^{(n)} = -{}^1/_{24} (1 - \xi_3^2)^2 \mathbf{M}_1 M_2' \mathbf{X}_2^{(n,1)} \qquad (n = 0, 1)$$
$$\mathbf{\Phi}^{(n)} = (\varphi_1^{(n)}, \varphi_2^{(n)}),$$
$$M_3' = \begin{Vmatrix} \partial_2^2 & \partial_1^2 & -\partial_1\partial_2 & 0 & 0 \\ 0 & 0 & 0 & \partial_2 & -\partial_1 \end{Vmatrix}$$

The vector $\mathbf{X}_1^{(n,0)}$ is written in the form

$$\mathbf{X}_1^{(n,0)} = \mathbf{X}_{1,0}^{(n,0)} + \mathbf{X}_{1,1}^{(n,0)}$$

Here the vector $\mathbf{X}_{1,0}^{(n,0)}$ represents a particular solution of an inhomogeneous vector equation of the system (2.5) in which $Q_\pm^{(n)} \equiv 0$, and the components of the vector $\mathbf{X}_{1,1}^{(n,0)}$ are given by

$$u_{1,1}^{(n,0)} = \int_0^{\xi_1} [(s_{12}^D\partial_1^2 - s_{16}^D\partial_1\partial_2 + s_{11}^D\partial_2^2)\varphi_1^{(n)} - \frac{1}{4\pi}(g_{21}\partial_1 - g_{11}\partial_2)\varphi_2^{(n)}]\, d\xi_1 + c^{(n)}\xi_2 + c_1^{(n)}$$

$$u_{2,1}^{(n,0)} = \int_0^{\xi_2} [(s_{22}^D\partial_1^2 - s_{26}^D\partial_1\partial_2 + s_{12}^D\partial_2^2)\varphi_1^{(n)} - \frac{1}{4\pi}(g_{22}\partial_1 - g_{12}\partial_2)\varphi_2^{(n)}]\, d\xi_2 - c^{(n)}\xi_1 + c_2^{(n)}$$

$$v_1^{(n,0)} = -\int_0^{\xi_1} [(g_{12}\partial_1^2 - g_{16}\partial_1\partial_2 + g_{11}\partial_2^2)\varphi_1^{(n)} + (\beta_{12}^t\partial_1 - \beta_{11}^t\partial_2)\varphi_2^{(n)}]\, d\xi_1 + c_3^{(n)}$$

where $c^{(n)}$, $c^{(n)}{}_1$, and $c^{(n)}{}_2$ are constants characterizing the rigid rotation and displacement of the plate in its middle surface, and $c_3^{(n)}$ denotes the zero potential of the electric field.

The system of differential equations for the functions $\varphi_1^{(n)}$, $\varphi_2^{(n)}$ and $u_3^{(n,0)}$ assumes the form

$$L_4\varphi_1^{(n)} + \frac{1}{4\pi}L_3\varphi_2^{(n)} = 0, \quad L_3\varphi_1^{(n)} - L_2\varphi_2^{(n)} = 0 \tag{3.2}$$

$$\mathbf{M}_1 M_2' B_{21}^{-1} M_2 \mathbf{M}_1 u_3^{(n,0)} = \mathbf{M}_1 M_2' B_{21}^{-1} M_2 B_{11} M_3 \mathbf{\Phi}^{(n-1)}$$

$$L_2 = \beta_{22}^t\partial_1^2 - 2\beta_{12}^t\partial_1\partial_2 + \beta_{11}^t\partial_2^2$$

$$L_3 = -g_{22}\partial_1^3 + (g_{12} + g_{26})\partial_1^2\partial_2 - (g_{21} + g_{16})\partial_1\partial_2^2 + g_{11}\partial_2^3$$

$$L_4 = s_{22}^D\partial_1^4 - 2s_{26}^D\partial_1^3\partial_2 + (2s_{12}^D + s_{66}^D)\partial_1^2\partial_2^2 - 2s_{16}^D\partial_1\partial_2^3 + s_{11}^D\partial_2^4$$

As before, we shall assume that the quantities in (3.1) and (3.2) accompanied by the index n vanish when $n < 0$.

In order to satisfy the boundary conditions on the edge surface of a thin piezoelectric plate, we must construct additional iterative processes describing boundary layer-type solutions [1]. The basic iterative process is however sufficient to fulfil these conditions in an integral manner.

We note that the first two equations of (3.2) coincide with the system of solution equations of the applied theory of generalized plane stressed state of thin piezoelectric plates constructed in [2,3] by averaging the fundamental electromechanical characteristics. The last equation of (3.2) coincides formally with the equation of the classical theory of flexure of thin anisotropic plates [6]. Both applied theories presume that the plate has a plane of material symmetry parallel to its middle surface.

4. Analyzing the relations (2.4), (2.5) and (3.1),(3.2), we can conclude that the absence of the plane of material symmetry parallel to the middle surface does not affect the construction of the initial iterative step of (2.3), and becomes manifest only in the derivation of the further approximation relations. Consequently the basic formulas of the generalized plane state of stress of a thin piezoelectric plate obtained in [2, 3] by averaging the electroelastic characteristics and under the assumption that a plane of material symmetry parallel to the middle surface exists, also hold for the thin piezoelectric plates with general anisotropy properties. The last argument is also valid for the anisotropic, classical-type plates. This can be shown by assuming that all piezoelectric moduli are equal to zero.

From (2.4) and (3.1) it follows that in the case of asymmetric loading of the piezoelectric layer or plate, the Kirchhoff hypotheses hold in the zeroth approximation and the hypothesis of the linearity of the electrostatic field introduced in [4] fails even in the zeroth approximation, irrespective of the presence of the plane of material symmetry.

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